

# On Sound Reflection in Superfluid

L.A. Melnikovsky\*

## Abstract

We consider reflection of the first and the second sound waves by a rigid flat wall in superfluid. Nontrivial dependence of the reflection coefficients on the angle of incidence is obtained. Sound conversion is predicted at slanted incidence.

## 1 Introduction

Bulk superfluid has twice as many variables as a normal fluid[1]. Direct consequence is the existence of two independent sound modes in helium II referred to as the first sound and the second sound. Due to anomalously small thermal expansion of helium, these modes can be viewed as purely pressure and temperature waves respectively. The “purity” is essential, *e.g.*, for the effectiveness of sound emission by various sources.

Restricted geometry effectively eliminates some of the hydrodynamic variables. Particularly, a steady cell wall eliminates the normal component of the mass flux  $\mathbf{j}$  and the normal velocity  $\mathbf{v}_n$ : the boundary condition at the wall is

$$\mathbf{j}_\perp = 0, \quad \mathbf{v}_n = 0. \quad (1)$$

The fourth sound [2] is one result of such elimination. This is the only sound mode in a narrow channel, both temperature and pressure oscillate coherently in this wave. In general, the sound modes, independent in bulk liquid, begin to interact at the boundary.\* This necessitates complete two-fluid consideration of the sound reflection in superfluid which is the subject of the present paper.

To avoid complications associated with the wall deformation consider a perfectly rigid flat wall. We therefore ignore numerous peculiarities of the sound transmission into solids and limit ourselves to linear hydrodynamic equations only. In Sec.2 we find all *three* nontrivial harmonic (*i.e.*, proportional to  $e^{i\mathbf{k}\mathbf{r}-i\omega t}$ ) solutions of the equations. It is important to mention that unbounded solutions (those with complex wave vector  $\mathbf{k}$ ) should not be overlooked in the restricted geometry.

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\*E-mail: leva@kapitza.ras.ru

\*Sound reflection at the free helium surface and at the solid helium boundary is extensively explored [3],[4], [5], and [6].

The boundary condition (1) selects a two-dimensional subspace of solutions for particular frequency  $\omega$ . Specific solutions correspond to the first sound reflection (Sec.3) and the second sound reflection (Sec.4). Interestingly enough, some part of the incident wave energy is transformed between the first and the second sound.

## 2 Harmonic solutions

Consider the linearized equations of superfluid hydrodynamics[8]:

$$\dot{\rho} + \frac{\partial j^i}{\partial x^i} = 0, \quad (2)$$

$$j^i + \frac{\partial p}{\partial x^i} = \eta \frac{\partial}{\partial x^k} \left( \frac{\partial v_n^i}{\partial x^k} + \frac{\partial v_n^k}{\partial x^i} - \delta^{ik} \frac{2}{3} \frac{\partial v_n^l}{\partial x^l} \right) + \frac{\partial}{\partial x^i} \left( \zeta_2 \frac{\partial v_n^l}{\partial x^l} - \zeta_1 \frac{\partial \rho_s w^l}{\partial x^l} \right), \quad (3)$$

$$v_s^k + \frac{\partial \mu}{\partial x^k} = \frac{\partial}{\partial x^k} \left( \zeta_4 \frac{\partial v_n^l}{\partial x^l} - \zeta_3 \frac{\partial \rho_s w^l}{\partial x^l} \right), \quad (4)$$

$$T \left( \dot{\sigma} \rho + \sigma \dot{\rho} + \sigma \rho \frac{\partial v_n^l}{\partial x^l} \right) = \kappa \frac{\partial^2 T}{\partial x^l \partial x^l}, \quad (5)$$

where  $\eta$ ,  $\zeta_1 = \zeta_4$ ,  $\zeta_2$ ,  $\zeta_3$ ,  $\kappa$  are dissipative coefficients,  $\sigma$  is entropy per unit mass,  $p$ ,  $\mu$ ,  $T$  are pressure, chemical potential, and temperature,  $\mathbf{v}_s$ ,  $\mathbf{v}_n$ , and  $\mathbf{w} = \mathbf{v}_n - \mathbf{v}_s$  are superfluid, normal, and relative velocities,  $\rho$  and  $\mathbf{j}$  are mass and momentum densities. The velocities and the momentum density are coupled by the equation

$$\mathbf{j} = \rho \mathbf{v}_s + \rho_n \mathbf{w} = \rho \mathbf{v}_n - \rho_s \mathbf{w}. \quad (6)$$

Further simplification is facilitated by ignoring thermal expansion (we therefore disregard the difference between specific heats  $c = T \partial \sigma / \partial T$  at constant pressure and at constant volume). Namely put

$$p' = s^2 \rho', \quad (7)$$

$$\rho \mu' = -\sigma \rho T' + p' = -\sigma \rho T' + s^2 \rho', \text{ and} \quad (8)$$

$$T \sigma' = c T', \quad (9)$$

where  $s = (\partial p / \partial \rho)^{1/2}$  is the first sound velocity. The prime denotes small deviation of the variables from their equilibrium values.

In a harmonic perturbation, space and time dependence of all deviations has a form of  $\exp(i\mathbf{k}\mathbf{r} - i\omega t)$ . To find all possible harmonic excitations in bulk superfluid we substitute this exponential term in Eqs.2-5 and keep linear terms only.

The mass conservation (2) gives

$$\omega \rho' = k^i j^i. \quad (10)$$

The momentum conservation law (3) can be transformed as follows:

$$-i\omega j^i + i p' k^i = -\eta k^k \left( v_n^i k^k + k^i v_n^k - \frac{2}{3} \delta^{ik} k^l v_n^l \right) - k^i k^k \left( -\rho_s \zeta_1 w^k + \zeta_2 v_n^k \right),$$

and, using (7), (6), and (10)

$$(\omega + i\eta k^2/\rho)j^i + (iA - s^2/\omega)k^i k^k j^k + i\eta k^2 \rho_s w^i/\rho + iB k^i k^k w^k = 0, \quad (11)$$

where the constants  $A = (\eta/3 + \zeta_2)/\rho$  and  $B = (A - \zeta_1)\rho_s$ .

From the energy conservation law (5) for harmonic deviation we get

$$T\omega\sigma'\rho + T\omega\sigma\rho' - T\sigma\rho v_n^l k^l + ik^2 \kappa T' = 0.$$

Using (6), (9), and (10) this can be reduced to

$$T\sigma\rho_s k^i w^i = (c\omega\rho + i\kappa k^2)T'. \quad (12)$$

Finally, substituting exponential term and (6) in (4) we obtain

$$-i\omega(j^i - \rho_n w^i)/\rho + ik^i \mu' = -k^i (-\rho_s \zeta_3 w^k k^k + \zeta_4 (j^k + \rho_s w^k) k^k/\rho)$$

Combining this with (8), (10), and (12)

$$\omega j^i + (i\zeta_4 - s^2/\omega)k^i k^k j^k - \omega\rho_n w^i + Gk^i k^k w^k = 0, \quad (13)$$

where

$$G \approx \frac{T\rho_s\sigma^2}{c\omega} - i\rho_s \left( \frac{T\kappa k^2\sigma^2}{c^2\omega^2\rho} - (\zeta_4 - \rho\zeta_3) \right)$$

Eqs. (11) and (13) can be written together as

$$\hat{L} \begin{pmatrix} \mathbf{j} \\ \mathbf{w} \end{pmatrix} = 0,$$

where  $\hat{L}$  is a square  $6 \times 6$  matrix composed of the coefficients from (11) and (13). The linear system is consistent if  $\det \hat{L} = 0$ . Due to the system isotropy, the determinant can not depend on individual components of  $k^i$ . Instead it depends on  $k^2 = k^i k^i$  only. We therefore can put  $k^y = k^z = 0$  and treat  $\hat{L}$  as a  $4 \times 4$  matrix:

$$\hat{L} = \begin{pmatrix} \omega + (-s^2/\omega + iA + i\eta/\rho)k^2 & 0 & i(\eta\rho_s/\rho + B)k^2 & 0 \\ 0 & \omega + i\eta k^2/\rho & 0 & i\eta k^2 \rho_s/\rho \\ \omega + (-s^2/\omega + i\zeta_4)k^2 & 0 & -\omega\rho_n + Gk^2 & 0 \\ 0 & \omega & 0 & -\omega\rho_n \end{pmatrix}$$

After factorization  $\det \hat{L}$  simplifies to

$$\begin{vmatrix} \omega + (iA + i\eta/\rho - s^2/\omega)k^2 & i(\eta\rho_s/\rho + B)k^2 \\ \omega + (i\zeta_4 - s^2/\omega)k^2 & -\omega\rho_n + Gk^2 \end{vmatrix} \cdot \begin{vmatrix} \omega + i\eta k^2/\rho & i\eta k^2 \rho_s/\rho \\ \omega & -\omega\rho_n \end{vmatrix}.$$

All nontrivial solutions immediately follow:

$$\omega^2 \approx k_1^2 \left( s^2 - i\omega A - i\frac{\omega\eta}{\rho} \right) \approx k_1^2 \left( s^2 - i\frac{\omega}{\rho} \left( \frac{4\eta}{3} + \zeta_2 \right) \right) \approx k_1^2 s^2, \quad (14)$$

$$\omega^2 \approx k_2^2 \omega_2 (G - i\frac{\eta\rho_s}{\rho} - iB)/\rho_n \approx k_2^2 \frac{T\rho_s\sigma^2}{c\rho_n} \equiv k_2^2 s_2^2, \quad (15)$$

$$\omega = -i\eta k_3^2/\rho_n, \quad (16)$$

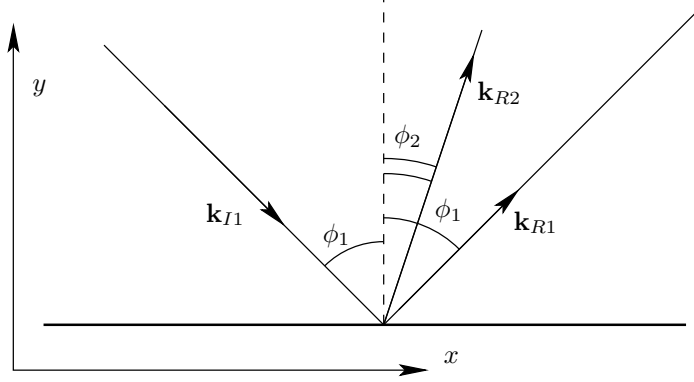


Figure 1: First sound reflection

where  $s_2$  is the second sound velocity.

Roots  $\mathbf{k}_1$  (14) and  $\mathbf{k}_2$  (15) correspond to “longitudinal” solutions where  $j^i \propto w^i \propto k_{1,2}^i$ , while the root  $\mathbf{k}_3$  (16) corresponds to a “transverse” one  $j^i k_3^i = w^i k_3^i = 0$ . The approximation in (14) and (15) is based on an assumption of low bulk damping, *i.e.*,  $|k_3| \gg |k_2| > |k_1|$ . This implies complete splitting between the first and second sound, namely  $\mathbf{w} = 0$  for (14) and  $\mathbf{j} = 0$  for (15). In the third solution (16), the superfluid velocity vanishes  $\mathbf{v}_s = 0$ , *i.e.*, the mass flux and the relative velocity are coupled by the relation  $\mathbf{j} = \rho_n \mathbf{w}$ .

### 3 First sound reflection

Consider the first sound wave incident towards the impervious rigid plane wall at an angle  $\phi_1$  (see Fig.1). The subscripts *I1*, *R1*, and *R2* refer to the incident first, reflected first, and reflected second sound waves respectively. The  $x$  axis runs along the wall and the  $y$  axis is directed into the liquid.

The heat transfer through the interface at low temperature can be neglected due to Kapitza resistance. Appropriate boundary conditions are  $j^y = 0$ ,  $w^y = 0$ ,  $j^x + \rho_s w^x = 0$ . They can be written in the matrix form

$$\begin{pmatrix} j_{I1} \sin \phi_1 \\ -j_{I1} \cos \phi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} j_{R1} \sin \phi_1 \\ j_{R1} \cos \phi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w_{R2} \sin \phi_2 \\ w_{R2} \cos \phi_2 \end{pmatrix} + \begin{pmatrix} \rho_n w_3^x \\ \rho_n w_3^y \\ w_3^x \\ w_3^y \end{pmatrix} = \begin{pmatrix} -\rho_s w \\ 0 \\ w \\ 0 \end{pmatrix}, \quad (17)$$

where  $\cos^2 \phi_2 = 1 - (s_2^2/s^2) \sin^2 \phi_1$ , to satisfy the condition  $k_{I1}^x = k_{R2}^x$ . The last term in the left-hand side of (17) represents the transverse surface wave with a wave vector  $\mathbf{k}_3$ . The wave must decay away from the boundary, therefore  $\text{Im } k_3^y > 0$ . This requirement selects the sign in (3), which is the transversality

relation  $\mathbf{w}_3 \perp \mathbf{k}_3$ :

$$\begin{pmatrix} w_3^x \\ w_3^y \end{pmatrix} \propto \begin{pmatrix} -k_3^y \\ k_3^x \end{pmatrix} \approx \begin{pmatrix} -k_3 \\ k_1 \sin \phi_1 \end{pmatrix} = \begin{pmatrix} -\varkappa e^{i\pi/4} \\ k_1 \sin \phi_1 \end{pmatrix} = \begin{pmatrix} -\varkappa e^{i\pi/4} \\ k_2 \sin \phi_2 \end{pmatrix},$$

where  $\varkappa = \sqrt{\omega \rho_n / \eta}$ . Substituting this in (17) we get

$$\begin{pmatrix} j_{I1} \sin \phi_1 \\ -j_{I1} \cos \phi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} j_{R1} \sin \phi_1 \\ j_{R1} \cos \phi_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_s w_{R2} \sin \phi_2 \\ 0 \\ w_{R2} \cos \phi_2 \end{pmatrix} + \begin{pmatrix} \rho w_3^x \\ \rho_n w_3^y \\ w_3^y \end{pmatrix} = 0$$

and

$$2j_{I1} + w_{R2} \left( \rho_s \frac{\sin \phi_2}{\sin \phi_1} + \rho_n \frac{\cos \phi_2}{\cos \phi_1} + e^{i\pi/4} \frac{\varkappa \rho \cos \phi_2}{k_1 \sin^2 \phi_1} \right) = 0$$

Second sound is slower than the first one  $s > s_2$ , consequently  $\phi_2 < \pi/2$  and  $\cos \phi_2 \neq 0$ . One can therefore neglect the first term in parenthesis

$$w_{R2} = -j_{I1} \frac{\sin^2 \phi_1}{\cos \phi_2} \frac{2k_1}{\rho_n k_1 \sin \phi_1 \tan \phi_1 + e^{i\pi/4} \varkappa \rho}. \quad (18)$$

Similarly, the amplitude of the reflected first sound is obtained from the equation

$$\begin{pmatrix} j_{I1} \rho_n k_1 \sin^2 \phi_1 \\ -j_{I1} \rho \varkappa e^{i\pi/4} \cos \phi_1 \end{pmatrix} + \begin{pmatrix} j_{R1} \rho_n k_1 \sin^2 \phi_1 \\ j_{R1} \rho \varkappa e^{i\pi/4} \cos \phi_1 \end{pmatrix} + \begin{pmatrix} \rho \rho_n k_1 \sin \phi_1 w_3^x \\ \rho \rho_n \varkappa e^{i\pi/4} w_3^y \end{pmatrix} = 0.$$

From this we have

$$\frac{j_{R1}}{j_{I1}} = \frac{\rho \varkappa \cos \phi_1 - \rho_n k_1 \sin^2 \phi_1 e^{-i\pi/4}}{\rho \varkappa \cos \phi_1 + \rho_n k_1 \sin^2 \phi_1 e^{-i\pi/4}}. \quad (19)$$

Reflection and conversion efficiency must be characterized by appropriate coefficients  $R_{11} = F_{R1}/F_{I1}$  and  $R_{12} = F_{R2}/F_{I1}$  respectively. Here  $F_1$  and  $F_2$  are the energy fluxes in the first and second sound waves. They are given by the expressions

$$F_1 = \frac{s}{2\rho} |j|^2, \quad F_2 = \frac{s_2 \rho_s \rho_n}{2\rho} |w|^2.$$

Using (19) and (18) we get

$$R_{11} = \left| \frac{\rho \varkappa \cos \phi_1 - \rho_n k_1 \sin^2 \phi_1 e^{-i\pi/4}}{\rho \varkappa \cos \phi_1 + \rho_n k_1 \sin^2 \phi_1 e^{-i\pi/4}} \right|^2, \quad (20)$$

$$R_{12} = \frac{s_2 \sin^4 \phi_1}{s \cos^2 \phi_2} \frac{4\rho_s \rho_n k_1^2}{|\rho_n k_1 \sin \phi_1 \tan \phi_1 + \rho \varkappa e^{i\pi/4}|^2}. \quad (21)$$

Sample graph of these functions is illustrated on Fig.2. The reflection coefficient  $R_{11}$  has a minimum of

$$\min R_{11} = 3 - 2\sqrt{2} \quad (22)$$

at finite angle of incidence. The value at the minimum is the same as for the sound reflection in usual hydrodynamics[7].

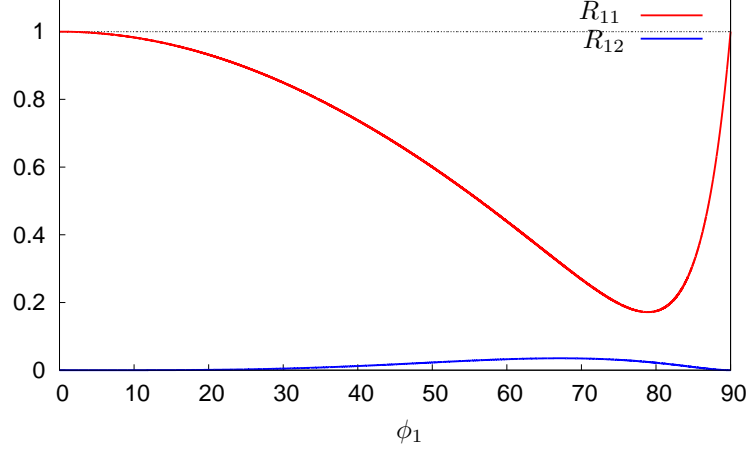


Figure 2: Reflection and conversion coefficients  $R_{11}$  and  $R_{12}$  vs. the angle of incidence  $\phi_1$

## 4 Second sound reflection

The same approach can be used to investigate the second sound wave incident at an angle  $\phi_2$  (see Fig.3). The boundary conditions in this case are

$$\begin{pmatrix} 0 \\ 0 \\ w_{I2} \sin \phi_2 \\ -w_{I2} \cos \phi_2 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ w_{R2} \sin \phi_2 \\ w_{R2} \cos \phi_2 \end{pmatrix} + \begin{pmatrix} j_{R1} \sin \phi_1 \\ j_{R1} \cos \phi_1 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} \rho_n w_3^x \\ \rho_n w_3^y \\ w_3^x \\ w_3^y \end{pmatrix} = \begin{pmatrix} -\rho_s w \\ 0 \\ w \\ 0 \end{pmatrix}.$$

After simplification this gives

$$2\rho_s \rho_n w_{I2} = j_{R1} \left( -\frac{\rho_n e^{i\pi/4} \cos \phi_1}{k_2 \sin^2 \phi_2} - \rho_n \frac{\sin \phi_1}{\sin \phi_2} - \rho_s \frac{\cos \phi_1}{\cos \phi_2} \right)$$

The last term in parenthesis is always negligible (last equation is meaningful only if  $\sin \phi_2 < s_2/s$ ). This gives

$$j_{R1} = -\rho_s w_{I2} \frac{2\rho_n k_2 \sin^2 \phi_2}{\rho_n e^{i\pi/4} \cos \phi_1 + \rho_n k_2 \sin \phi_1 \sin \phi_2}. \quad (23)$$

The conversion coefficient  $R_{21} = F_{R1}/F_{I2}$  is therefore given by

$$R_{21} = \frac{4s}{s_2} \frac{\rho_s \rho_n k_2^2 \sin^4 \phi_2}{|\rho_n e^{i\pi/4} \cos \phi_1 + \rho_n k_2 \sin \phi_1 \sin \phi_2|^2}. \quad (24)$$

Its maximum

$$\max R_{21} = 4 \frac{\rho_s s_2}{\rho_n s} \quad (25)$$

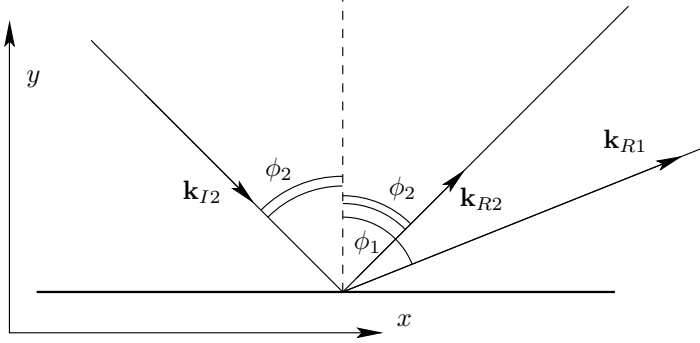


Figure 3: Second sound reflection

is reached at the critical angle  $\sin \phi_2 = s_2/s$ .

Amplitude of the reflected second sound wave is found from the relation

$$\frac{w_{R2}}{w_{I2}} \left( \frac{\kappa \rho e^{i\pi/4}}{k_2 \sin \phi_2} + \rho_n \tan \phi_1 + \rho_s \tan \phi_2 \right) = \left( \frac{\kappa \rho e^{i\pi/4}}{k_2 \sin \phi_2} + \rho_n \tan \phi_1 - \rho_s \tan \phi_2 \right), \quad (26)$$

where  $\tan \phi_1 = s \sin \phi_2 / \sqrt{s_2^2 - s^2 \sin^2 \phi_2}$  and  $\text{Im} \tan \phi_1 \leq 0$  (selected by the requirement  $\text{Im} k_1^y \geq 0$ ). The reflection coefficient is therefore

$$R_{22} = \left| \frac{\rho \kappa e^{i\pi/4} + \rho_n k_2 \sin \phi_2 \tan \phi_1 - \rho_s k_2 \sin \phi_2 \tan \phi_2}{\rho \kappa e^{i\pi/4} + \rho_n k_2 \sin \phi_2 \tan \phi_1 + \rho_s k_2 \sin \phi_2 \tan \phi_2} \right|^2. \quad (27)$$

These functions for sample parameters are plotted on Fig.4.

## 5 Discussion

It is shown that sound reflection at slanted incidence by a plane impervious wall is suppressed for both first and second sound. This phenomenon is similar to Konstantinov effect in ordinary gases.

Coincidentally with the reflection suppression a sound conversion takes place. The effect is predicted to have strong angle dependence and should allow experimental verification. Moreover, there is a number of the heat pulse propagation measurements (*e.g.*, [9] and [10]) where the pulse transit time was often much shorter than that for the second sound. This phenomenon is usually explained by anomalously long phonon free path at low temperatures or by sound conversion in bulk (due to nonlinear effects) or at liquid-vapour interface. It seems probable that fast propagation is in fact the manifestation of the sound conversion described in this paper, so that the heat pulse is transformed at some wall into the pressure pulse and is later transformed back near the receiver. The

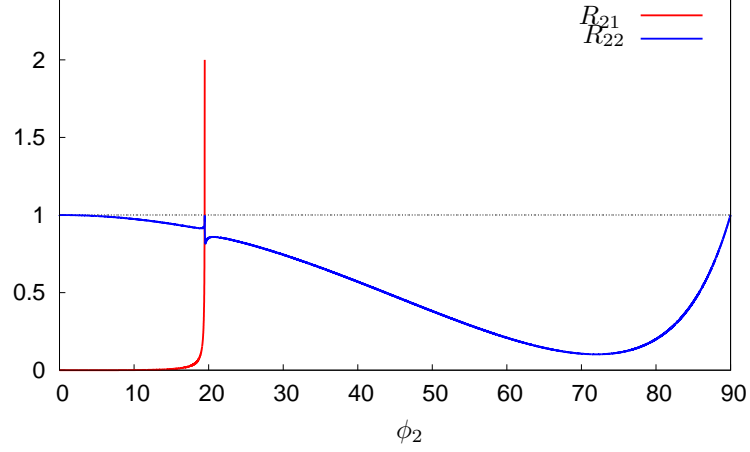


Figure 4: Reflection and conversion coefficients  $R_{22}$  and  $R_{21}$  vs. the angle of incidence  $\phi_2$ .

signal therefore travels (some part of) the path with the velocity of the first sound.

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## References

- [1] L.D.Landau, *J. Phys. USSR* **5**, 71 (1941).
- [2] K.R.Atkins, *Phys. Rev.* **113**, 962 (1959).
- [3] J.R.Pellam, *Phys. Rev.* **73**, 608 (1948).
- [4] R.B.Dingle, *Proc. Phys. Soc. A* **61**, 9 (1948).
- [5] D.M.Chernikova, *Sov. Phys. JETP* **20**, 358 (1965).
- [6] M.Yu.Kagan, Yu.A.Kosevich, *Sov. J. Low Temp. Phys.* **14**, 433 (1988).
- [7] B.P.Konstantinov, *Zh. Tekh. Fiz.* **9**, 226 (1939).
- [8] I.M.Khalatnikov, *An Introduction to the theory of superfluidity* (W.A.Benjamin, New York-Amsterdam 1965).



- [9] J.R.Pellam, *Phys. Rev.* **75**, 1183 (1949).
- [10] N.Mulders, R.Mehrotra, L.S.Goldner, G.Alles, *Phys. Rev. Lett.* **67**, 695 (1991).